(a) $f_x(x,y,z) = 2xz + y^2$ implies $f(x,y,z) = x^2z + xy^2 + g(y,z)$ and so $f_y(x,y,z) = 2xy + g_y(y,z)$. But $f_y(x,y,z) = 2xy$ so $g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$.
Thus $f(x,y,z) = x^2z + xy^2 + h(z)$ and $f_z(x,y,z) = x^2 + h'(z)$. But $f_z(x,y,z) = x^2 + 6z^2$, so $h'(z) = 6z^2 \quad \Rightarrow$
$h(z) = 2z^3 + K$.
Hence $f(x,y,z) = x^2z + xy^2 + 2z^3$ (taking $K = 0$).

(b) $t = 0$ corresponds to the point $(0,2,-1)$ and $t = 1$ corresponds to $(1,3,3)$, so
$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,3,3) - f(0,2,-1) = 66 - -2 = 68$.