Maximize $f(x, y) = \dfrac{xy}{3}(5 - x - 2y)$, then the maximum volume is $V = xyz$.

$f_x = \frac{1}{3}(5y - 2xy - y^2) = \frac{1}{3}y(5 - 2x - 2y)$ and $f_y = \frac{1}{3}x(5 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives the critical point

$\left(\frac{5}{3}, \frac{5}{6}\right)$ which geometrically must yield a maximum. Thus the volume of the largest such box is $V = \left(\frac{5}{3}\right)\left(\frac{5}{6}\right)\left(\frac{5}{9}\right) = \frac{125}{162}$.